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An iterative method of global convergence without derivatives in the class of smooth functions *

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Abstract

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In the class of smooth functions, the iterative methods using only first-order derivatives or values of f are established by the idea introducing a parametric function and making the global estimate for the interpolating remainder. These methods are globally convergent and contain a real parameter λ ($\geqslant 0$). When $0 \leqslant \lambda \leqslant 1$, the order of convergence of the methods is $1+\lambda$ for a simple real zero of $f(x)$, and 1 for a multiple real zero. When $1 < \lambda$, these methods are only linearly convergent for any real zero of $f(x)$.

Keywords: Order of convergence; difference quotient; remainder term; global convergence.

1. Introduction

In order to solve the nonlinear equation $f(x)=0$, neither the secant method proposed in [8] nor the iterative methods proposed in [4,5] include derivatives when f belongs to the class of smooth functions, yet they are only locally convergent. The iterative methods involving higher-order derivatives proposed in [1–3,6,7] are globally convergent when f belongs to the class of the entire analytic functions of order not exceeding 2 and includes only real zeros. When $f(x)$ belongs to the class of smooth functions, that is, $f(x) \in C^1(\mathbb{R}^1)$, $\mathbb{R}^1 = (-\infty, +\infty)$, the iterative methods of global convergence without higher-order derivatives have been derived in [9].

In this paper the iterative methods using only first-order derivatives or values of f are established by the idea in [9,10] introducing a parametric function and making the global

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estimate for the interpolating remainder. These methods are globally convergent and contain a real parameter λ ($\geqslant 0$). When $0 \leqslant \lambda \leqslant 1$, the order of convergence of these methods is $1+\lambda$ for a simple real zero of $f(x)$, and 1 for a multiple real zero. These methods are only linearly convergent for a real zero of $f(x)$ when $1 < \lambda$.

2. The establishment of the iterative method with global convergence

Assume $f(x) \in C^1(\mathbb{R}^1)$, $\mathbb{R}^1 = (-\infty, +\infty)$. The real zero of $f(x)$ is denoted by $\bar{x}$. We form the Taylor expansion for $f(\bar{x})$ about x_n as follows:

$$f(\bar{x}) = f(x_n) + f^1(\xi_n)(\bar{x} - x_n) = 0, \quad \xi_n = x_n + \theta(\bar{x} - x_n),\ 0 < \theta < 1. \tag{2.1}$$

By (2.1) we obtain if $x_n \neq \bar{x}$,

$$f'(\xi_n) = \frac{-f(x_n)}{(\bar{x} - x_n).} \tag{2.2}$$

A parametric function $\phi(x)$ is introduced in the following form:

$$\phi(x) = \psi(x)\,|\,f(x)\,|, \quad \psi(x) > 0, \text{ as } x \in \mathbb{R}^1, \tag{2.3}$$

where the real parameter $\lambda \geqslant 0$ and given positive $\psi(x)$ is a piecewise smooth function with the discontinuity of the first kind. Let $\hat{\phi} = \hat{\phi}(x)$ be another parametric function with the above-mentioned property of $\phi(x)$. By the mean value theorem we have

$$f\big(x_n \pm \hat{\phi}\big) - f(x_n) = f'\big(x_n \pm \bar{\theta}\hat{\phi}\big)\big(\pm\hat{\phi}\big), \quad 0 < \bar{\theta} < 1. \tag{2.4}$$

Setting $\hat{\phi} = t\phi(x)/\bar{\theta}$ in (2.4) gives the difference quotient

$$f\left[x_n \pm t\frac{\phi}{\bar{\theta}},\ x_n\right] = f'(x_n \pm t\phi), \quad t \in [0, 1]. \tag{2.5}$$

Lemma 2.1. *Assume $f(x) \in C^1(\mathbb{R}^1)$, the sequence $\{x_n\}$ converges to the real zero $\bar{x}$ of $f(x)$ monotonously and the parametric function $\phi(x)$ is defined by* (2.3); *then the two following conclusions hold.*

(I) *If $f'(t)$ is monotone on intervals $x_n - \phi(x_n) \leqslant t \leqslant x_n$ and $x_n \leqslant t \leqslant x_n + \phi(x_n)$ when $|x_n - \bar{x}| < \phi(x_n)$, respectively, then we have the estimate*

$$\begin{aligned}
|f'(\xi_n)| &\leqslant \max\left(\frac{|f(x_n)|}{\phi(x_n)},\ |f'(x_n)|,\ \left|f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\ x_n\right]\right|\right) \equiv \hat{q}_1(x_n)\\
&\leqslant \max\left(\frac{|f(x_n)|}{\phi(x_n)} + |f'(x_n)|,\ \left|f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\ x_n\right]\right|\right) \equiv \hat{q}_2(x_n),\\
&x_n \in \mathbb{R}^1,
\end{aligned} \tag{2.6}$$

where signs + and − correspond to the increasing sequence $\{x_n^+\}$ and the decreasing sequence $\{x_n^-\}$, respectively; ϵ satisfies

$$0<\epsilon \leqslant \tfrac{1}{2}(\sqrt{5}-2). \tag{2.7}$$

(II) *If $f'(t)$ is monotone on the interval $|t-x_n| \leqslant \phi(x_n)$ when $|x_n-\bar{x}|<\phi(x_n)$, then we obtain the estimate*

$$\begin{aligned}|f'(\xi_n)| &\leqslant \max\left(\frac{|f(x_n)|}{\phi(x_n)}, \left|f\left[x_n-\frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\, x_n\right]\right|, \left|f\left[x_n+\frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\, x_n\right]\right|\right)\\ &\equiv q_1^*(x_n)\\ &\leqslant \max\left(\frac{|f(x_n)|}{\phi(x_n)} + \left|f\left[x_n\mp\frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\, x_n\right]\right|, \left|f\left[x_n\pm\frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\, x_n\right]\right|\right)\\ &\equiv q_2^*(x_n), \quad x_n \in \mathbb{R}^1. \end{aligned} \tag{2.8}$$

Proof. (I) Using Taylor's formula and the hypotheses, it is not difficult to prove that θ in (2.1) and $\bar{\theta}$ in (2.4) satisfy

$$\lim_{x_n\to\bar{x}} \theta = \tfrac{1}{2}, \quad \lim_{x_n\to\bar{x}} \bar{\theta} = \tfrac{1}{2}. \tag{2.9}$$

From (2.2) we obtain when $|x_n-\bar{x}| \geqslant \phi(x_n)$ that

$$|f'(\xi_n)| \leqslant \frac{|f(x_n)|}{\phi(x_n)}. \tag{2.10}$$

It is clear that $|\xi_n-x_n| = \theta|\bar{x}-x_n| < \theta\phi$ by the expression of ξ_n in (2.1) when $|x_n-\bar{x}|<\phi(x_n)$. This indicates that ξ_n lies in the interior of the interval $|t-x_n| = \theta|\bar{x}-x_n| < \theta\phi$. We now consider two separate cases for $\{x_n\}$.

Case I. Let $\{x_n\}$ be an increasing sequence $\{x_n^+\}$. In this case we have $0<\xi_n^+-x_n^+ = \theta(\bar{x}-x_n^+) < \theta\phi(x_n^+)$ by $|\bar{x}-x_n^+| < \phi(x_n^+)$ and the hypothesis. Thus, for any $\epsilon\in(0, \tfrac{1}{2}(\sqrt{5}-2)]$ we can choose a small enough (in absolute value) $\phi(x)$ such that

$$\theta, \bar{\theta} \in \left[\tfrac{1}{2}-\epsilon, \tfrac{1}{2}+\epsilon\right],$$

in view of (2.9) and $|\bar{x}-x_n|<\phi(x_n)$. And hence

$$\frac{\theta}{\bar{\theta}} \in \left[\frac{1-2\epsilon}{1+2\epsilon}, \frac{1+2\epsilon}{1-2\epsilon}\right]. \tag{2.11}$$

And it follows from (2.11) that

$$\theta = \frac{\theta}{\bar{\theta}}\bar{\theta} \leqslant \frac{1+2\epsilon}{1-2\epsilon}\bar{\theta} \leqslant \frac{(1+2\epsilon)^2}{2(1-2\epsilon)} \leqslant 1, \quad \text{as } 0<\epsilon\leqslant\tfrac{1}{2}(\sqrt{5}-2), \tag{2.12}$$

where the property that $(1+2\epsilon)^2/2(1-2\epsilon)$ is an increasing function on the interval $0<\epsilon \leqslant \tfrac{1}{2}(\sqrt{5}-2)$ is used.

On the other hand, if $f'(t)$ is an increasing function on the interval $x_n^+ \leqslant t \leqslant x_n^+ + \phi(x_n^+)$ when $|x_n^+-\bar{x}| < \phi(x_n^+)$, it follows from (2.5) and (2.12) that

$$-|f'(x_n^+)| < f'(\xi_n^+) \leqslant \left| f'\left(x_n^+ + \frac{1+2\epsilon}{1-2\epsilon} \bar{\theta} \phi \right) \right| = \left| f\left[x_n^+ + \frac{1+2\epsilon}{1-2\epsilon} \phi(x_n^+),\ x_n^+ \right] \right|. \tag{2.13}$$

If $f'(t)$ is a decreasing function on the interval $x_n^+ \leqslant t \leqslant x_n^+ + \phi(x_n^+)$ when $|x_n^+ - \bar{x}| \leqslant \phi(x_n^+)$, similarly, we then obtain in view of (2.5) and (2.12) that

$$-\left| f\left[x_n^+ + \frac{1+2\epsilon}{1-2\epsilon} \phi,\ x_n^+ \right] \right| \leqslant f'\left(x_n^+ + \frac{1+2\epsilon}{1-2\epsilon} \bar{\theta} \phi \right) \leqslant f'(\xi_n^+) < |f'(x_n^+)|. \tag{2.14}$$

Combination of the estimates (2.13) and (2.14) gives for any monotonic $f'(t)$ on $x_n^+ \leqslant t \leqslant x_n^+ + \phi(x_n^+)$ when $|x_n^+ - \bar{x}| < \phi(x_n^+)$ that

$$|f'(\xi_n^+)| \leqslant \max\left(|f'(x_n)|,\ \left| f\left[x_n^+ + \frac{1+2\epsilon}{1-2\epsilon} \phi(x_n^+),\ x_n^+ \right] \right| \right), \quad \text{for } 0 < \bar{x} - x_n^+ < \phi(x). \tag{2.15}$$

Case II. Let $\{x_n\}$ be a decreasing sequence $\{x_n^-\}$. In this case we have $-\theta\phi(x_n^-) < \theta(\bar{x} - x_n^-) = \xi_n^- - x_n^- < 0$ by $|x_n^- - \bar{x}| < \phi(x_n^-)$ and the hypothesis. Similarly, since $f'(t)$ is monotone on $x_n^- - \phi(x_n^-) \leqslant t \leqslant x_n^-$ when $|x_n^- - \bar{x}| < \phi(x_n^-)$, it follows from (2.5) and (2.12) that

$$|f'(\xi_n^-)| \leqslant \max\left(|f'(x_n^-)|,\ \left| f\left[x_n^- - \frac{1+2\epsilon}{1-2\epsilon} \phi(x_n^-),\ x_n^- \right] \right| \right), \quad \text{for } -\phi < \bar{x} - x_n^- < 0. \tag{2.16}$$

Thus, the expressions (2.15) and (2.16) concerning $\{x_n^+\}$ and $\{x_n^-\}$ can jointly be written in the form

$$|f'(\xi_n)| \leqslant \max\left(|f'(x_n)|,\ \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon} \phi(x_n),\ x_n \right] \right| \right), \quad \text{for } 0 \leqslant |x_n - \bar{x}| < \theta. \tag{2.17}$$

Clearly, (2.17) also holds for $x_n = \bar{x}$. Thus, (2.6) is proved by (2.10) and (2.17).

(II) In view of (2.5), (2.12) and the monotonicity of $f'(t)$ on $|t - x_n| \leqslant \phi(x_n)$ when $|x_n - \bar{x}| < \phi(x_n)$, we obtain when $f'(t)$ is increasing that

$$-\left| f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon} \phi,\ x_n \right] \right| \leqslant f'\left(x_n - \frac{1+2\epsilon}{1-2\epsilon} \bar{\theta} \phi \right) < f'(\xi_n) \leqslant \left| f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon} \phi,\ x_n \right] \right|, \tag{2.18}$$

and when $f'(t)$ is decreasing that

$$-\left| f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon} \phi,\ x_n \right] \right| \leqslant f'\left(x_n + \frac{1+2\epsilon}{1-2\epsilon} \bar{\theta} \phi \right) < f'(\xi_n) \leqslant \left| f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon} \phi,\ x_n \right] \right|. \tag{2.19}$$

Thus, combination of the estimates (2.18) and (2.19) gives

$$|f'(\xi_n)| \leqslant \max\left(\left|f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|,\ \left|f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|\right),$$
$$\text{for } |x_n - \bar{x}| < \phi(x_n). \tag{2.20}$$

The estimate (2.8) is obtained by (2.10) and (2.20). Thus, the lemma is proved. □

Remark 2.2. The condition in the lemma that $f'(t)$ is monotone on $x_n - \phi \leqslant t \leqslant x_n$, $x_n \leqslant t \leqslant x_n + \phi$, and $|t - x_n| \leqslant \phi$ when $|x_n - \bar{x}| < \phi$ respectively, is easily satisfied by choosing a sufficiently small $\phi(x)$ in absolute value. For example, the required $\phi(x)$ can be chosen as

$$\phi(x) = T\left(\frac{|f(x)|}{\nu + |f(x)|}\right)^{\lambda}, \quad \epsilon_0 \leqslant T,\ 1 \leqslant \nu,\ \lambda \geqslant 0, \tag{2.21}$$

where T, ν and λ are any given parameters, ϵ_0 is a pre-assigned accuracy of finding roots. It is obvious that the values of $|f(x_n)|$ are already enough small in the later phase of the iterative process, whereas we can choose λ and ν so large and T so small that $\phi(x)$ is still sufficiently small in the iterative initial phase. Thus, the condition that $f'(t)$ is monotone on the above intervals holds automatically when $\phi(x)$ is small enough. Hence, it is not a severe restriction for $f(x)$ that we impose a monotonic condition on $f'(t)$ on the intervals above.

Evidently, from (2.6) and (2.8) we have

$$|f'(\xi_n)| \leqslant \max\left(\frac{|f(x_n)|}{\phi(x_n)},\ |f'(x_n)| + \left|f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|\right) \equiv \hat{q}_3(x_n), \quad x_n \in \mathbb{R}^1, \tag{2.22}$$

and

$$|f'(\xi_n)| \leqslant \max\left(\frac{|f(x_n)|}{\phi(x_n)},\ \left|f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right| + \left|f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|\right)$$
$$= q_3^*(x_n), \quad x_n \in \mathbb{R}^1. \tag{2.23}$$

Noting $\hat{q}(x_n)$ is one of $\hat{q}_i(x_n)$, $i = 1, 2, 3$, $q^*(x_n)$ is one of $q_i^*(x_n)$, $i = 1, 2, 3$. From (2.2), (2.6), (2.8), (2.22) and (2.23) we then obtain inequalities

$$-\frac{\hat{q}(x_n)}{|f(x_n)|} \leqslant \frac{1}{\bar{x} - x_n} \leqslant \frac{\hat{q}(x_n)}{|f(x_n)|} \tag{2.24}$$

and

$$-\frac{q^*(x_n)}{|f(x_n)|} \leqslant \frac{1}{\bar{x} - x_n} \leqslant \frac{q^*(x_n)}{|f(x_n)|}. \tag{2.25}$$

In order to solve a nonlinear equation $f(x) = 0$, two iterative formulas can be defined as follows using the inequalities (2.24) and (2.25), respectively:

$$x_{n+1} = x_n \pm \frac{|f(x_n)|}{\hat{q}(x_n)}, \quad n = 0, 1, \ldots, \tag{2.26}$$

and

$$x_{n+1}=x_n \pm \frac{|f(x_n)|}{q^*(x_n)}, \quad n=0, 1,\ldots, \tag{2.27}$$

where the + and − signs in $\hat{q}(x_n)$ and $q^*(x_n)$ must agree with the ones in (2.26) and (2.27), respectively. Taking the signs + and − in (2.26) or (2.27) gives the increasing sequence $\{x_n^+\}$ and decreasing sequence $\{x_n^-\}$, respectively.

3. The global convergence and the order of convergence of the methods

Using the inequalities (2.24) and (2.25) respectively, we can prove that iterative formulas (2.26) and (2.27) possess the property of global convergence for $f \in C^1(\mathbb{R}^1)$.

Theorem 3.1. *Assume that $f(x) \in C^1(\mathbb{R}^1)$, $f'(t)$ is monotone on the interval $|t-x_n| \leqslant \phi(x_n)$ when $|x_n-\bar{x}| < \phi(x_n)$ and all real zeros of $f(x)$ are arranged in order as $\bar{x}_1 < \bar{x}_2 < \cdots$. Given the initial value x_0 is any real number such that $f(x_0) \neq 0$ and the parametric function $\phi(x)$ is defined by* (2.3), *then there are the following two conclusions for the monotonic sequences $\{x_n^+\}$ and $\{x_n^-\}$ generated by the iterative formula* (2.27).

(I) *If $\bar{x}_\nu < x_0 < \bar{x}_{\nu+1}$, where $\bar{x}_\nu$ and $\bar{x}_{\nu+1}$ are any neighboring two real zeros of $f(x)$, then the sequences $\{x_n^+\}$ and $\{x_n^-\}$ will converge monotonously increasing and decreasing to $\bar{x}_{\nu+1}$ and x_ν, respectively.*

(II) *If $\bar{x}_\nu < x_0$ (or $x_0 < \bar{x}_\nu$), $\nu = 1, 2,\ldots,$ then $\{x_n^-\}$ (or $\{x_n^+\}$) will converge monotonously decreasing (or increasing) to such a real zeros of $f(x)$ that is closest to the initial value x_0, and another sequence $\{x_n^+\}$ (or $\{x_n^-\}$) will diverge to $+\infty$ (or $-\infty$).*

Proof. (I) Since $f(x_0) \neq 0$, it follows from the iterative formula (2.27) that

$$x_1^{\pm} - x_0 = \pm \frac{|f(x_0)|}{q^*(x_0)} \begin{cases} >0, & \text{taking signs } +, \\ <0, & \text{taking signs } -. \end{cases} \tag{3.1}$$

Setting $n=0$ in (2.25) gives by using (3.1)

$$\frac{1}{x_1^- - x_0} \leqslant \frac{1}{\bar{x}-x_0} \leqslant \frac{1}{x_1^+ - x_0}. \tag{3.2}$$

Setting $\bar{x}=\bar{x}_\nu$ in (3.2) when $\bar{x}_\nu < x_0$ and $\bar{x}=\bar{x}_{\nu+1}$ in (3.2) when $x_0 < \bar{x}_{\nu+1}$ gives

$$\bar{x}_\nu \leqslant x_1^- < x_0 < x_1^+ \leqslant \bar{x}_{\nu+1}.$$

Applying mathematical induction we obtain

$$\bar{x}_\nu \leqslant \cdots < x_n^- < \cdots < x_2^- < x_1^- < x_0 < x_1^+ < x_2^+ < \cdots < x_n^+ < \cdots \leqslant \bar{x}_{\nu+1}.$$

By the criterion that a bounded monotone sequence must have a limit, we have

$$\lim_{n\to\infty} x_n^- = x^- \geqslant x_\nu^-, \qquad \lim_{n\to\infty} x_n^+ = x^+ \leqslant \bar{x}_{\nu+1}. \tag{3.3}$$

Noting that there exists a finite limit $\lim_{n\to\infty} q^*(x_n^\pm)$, it then follows from (3.3) and the iterative formula (2.27) that

$$\pm |f(x^\pm)| = \lim_{n\to\infty} q^*(x_n^\pm)(x_{n+1}^\pm - x_n^\pm) = 0. \tag{3.4}$$

Thus, in view of (3.3), (3.4) and the hypotheses we obtain

$$x^+ = \bar{x}_\nu, \qquad x^+ = \bar{x}_{\nu+1}.$$

This shows that the sequence $\{x_n^+\}$ (or $\{x_n^-\}$) must converge monotonously to such a real zero of $f(x)$ that is closest to x_0, provided that a real zero of $f(x)$ exists on the side to which the monotonic sequence $\{x_n^+\}$ (or $\{x_n^-\}$) tends. Conversely, the limit of $\{x_n^+\}$ (or $\{x_n^-\}$) must be a real zero of $f(x)$ if the monotone sequence $\{x_n^+\}$ (or $\{x_n^-\}$) is convergent.

(II) We know from the hypotheses that for $\{x_n^+\}$ (or $\{x_n^-\}$) we have $x_0 < x_1^+ < x_2^+ < \cdots < x_n^+ < \cdots$ (or $\cdots < x_n^- < \cdots < x_2^- < x_1^- < x_0$), this means that it is impossible that the monotonic sequence $\{x_n^+\}$ (or $\{x_n^-\}$) has a finite upper bound (or lower bound), otherwise it will contradict the hypothesis that all real zeros $\bar{x}_\nu$, $\nu = 1, 2, \ldots$, of $f(x)$ lie on one side of x_0. Hence, it follows that $\lim_{n\to\infty} x_n^+ = +\infty$ (or $\lim_{n\to\infty} x_n^- = -\infty$). On the other hand, as shown in (I), we can prove that $\{x_n^-\}$ (or $\{x_n^+\}$) will converge monotonously decreasing (or increasing) to such a real zero of $f(x)$ that is closest to the initial value x_0. The proof is complete. □

Theorem 3.2. *If $f(x) \in C^1(\mathbb{R}^1)$, $\phi(x)$ is also determined by* (2.3) *and $f'(t)$ is monotone on the intervals $x_n - \phi(x_n) \leqslant t \leqslant x_n$ and $x_n \leqslant t \leqslant x_n + \phi(x_n)$ when $|x_n - \bar{x}| < \phi(x_n)$, respectively, then the conclusions in Theorem* 3.1 *hold for the sequences $\{x_n^+\}$ and $\{x_n^-\}$ generated by the iterative formula* (2.26).

Theorem 3.2 can be proved using inequality (2.24) in a way similar to the proof of Theorem 3.1. Thus, the proof may be omitted.

Theorem 3.3. *Assume that the parametric function $\phi(x)$ is defined by* (2.3), *$f'(t)$ is monotone on $|t - x_n| \leqslant \phi(x_n)$ when $|x_n - \bar{x}| < \phi(x_n)$ and $f(x) \in C^{k+1}(\mathbb{R}^1)$, where k ($\geqslant 1$) is the multiplicity of the real zero $\bar{x}$ of $f(x)$. Taking $q^*(x_n) = q_1^*(x_n)$, there are the following three conclusions for the sequence $\{x_n\}$ generated by the iterative formula* (2.27) *for different values of the parameter λ.*

(I) *Let $\lambda \in (0, 1)$ in $\phi(x)$; then the order of convergence of $\{x_n\}$ is $1 + \lambda$ for $k = 1$, and 1 for $k \geqslant 2$.*

(II) *Let $\lambda = 0$ or $\lambda > 1$ in $\phi(x)$; then the order of convergence of $\{x_n\}$ is 1 for $k \geqslant 1$.*

(III) *Let $\lambda = 1$ in $\phi(x)$ and let δ_0 be positive and sufficiently small; then it follows that*

(a) *the order of convergence of $\{x_n\}$ is 2 for $k = 1$, and 1 for $2 \leqslant k$, if*

$$\frac{1}{\psi(x_n)} \leqslant \max\left(\left|f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\ x_n\right]\right|,\ \left|f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\ x_n\right]\right|\right), \tag{3.5}$$

in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$;

(b) *the order of convergence of $\{x_n\}$ is 1 for $1 \leqslant k$, if*

$$\frac{1}{\psi(x_n)} > \max\left(\left|f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|,\ \left|f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|\right), \tag{3.6}$$

in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$.

Proof. Take the initial value x_0 to be any real number such that $f(x_0) \neq 0$. Since $f(x) \in C^{k+1}(\mathbb{R}^1)$, we now form the Taylor expansion of order 2 for $f(\bar{x})$ about x_n as follows:

$$f(\bar{x}) = f(x_n) + f'(x_n)(\bar{x} - x_n) + \tfrac{1}{2}f''(\zeta_n)(\bar{x} - x_n)^2 = 0, \tag{3.7}$$

where $\zeta_n = x_n + \theta^*(\bar{x} - x_n)$, $0 < \theta^* < 1$. Using the iterative formula (2.27) we obtain on dividing (3.7) by $q_1^*(x_n)$ that

$$e_{n+1} = \begin{cases} \dfrac{q_1^*(x_n) \pm \left(f'(x_n) - \frac{1}{2}f''(\zeta_n)e_n\right)}{q_1^*(x_n)} e_n, & \text{for } f(x_n) > 0, \\ \dfrac{q_1^*(x_n) \pm \left(-f'(x_n) + \frac{1}{2}f''(\zeta_n)e_n\right)}{q_1^*(x_n)} e_n, & \text{for } f(x_n) < 0, \end{cases} \tag{3.8}$$

where $e_n = x_n - \bar{x}$, and the signs $\pm$ correspond to sequences $\{x_n^+\}$ and $\{x_n^-\}$, respectively. Theorem 3.1 shows that the iterative sequence $\{x_n\}$ generated by (2.27) will converge monotonously to a real zero $\bar{x}$ of $f(x)$ if there exists a real zero $\bar{x}$ of $f(x)$. Hence, we can assert when x_n lies in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ (or when n is large enough) that

$$(x_n - \bar{x})f(x_n)f'(x_n) > 0, \quad \text{when } 0 < |x_n - \bar{x}| \leqslant \delta_0. \tag{3.9}$$

(I) For $K = 1$ we form the Taylor expansions for

$$f\left(x_n^\pm + \frac{1+2\epsilon}{1-2\epsilon}\phi\right)$$

and

$$f\left(x_n^\pm - \frac{1+2\epsilon}{1-2\epsilon}\phi\right)$$

about x_n as follows:

$$f\left(x_n^\pm + \frac{1+2\epsilon}{1-2\epsilon}\phi\right) = f(x_n^\pm) + f'(x_n^\pm)\frac{1+2\epsilon}{1-2\epsilon}\phi + \tfrac{1}{2}f''(\hat{\eta}_n^\pm)\left(\frac{1+2\epsilon}{1-2\epsilon}\phi\right)^2, \tag{3.10}$$

where

$$\hat{\eta}_n^\pm = x_n^\pm + \hat{\theta}^\pm \frac{1+2\epsilon}{1-2\epsilon}\phi(x_n^\pm)|f(x_n^\pm)|^\lambda, \quad 0 < \hat{\theta}^\pm < 1,$$

and

$$f\left(x_n^\pm - \frac{1+2\epsilon}{1-2\epsilon}\phi\right) = f(x_n^\pm) - f'(x_n^\pm)\frac{1+2\epsilon}{1-2\epsilon}\phi + \tfrac{1}{2}f''(\tilde{\eta}_n^\pm)\left(\frac{1+2\epsilon}{1-2\epsilon}\phi\right)^2, \tag{3.11}$$

where

$$\tilde{\eta}_n^\pm = x_n^\pm - \tilde{\theta}^\pm \frac{1+2\epsilon}{1-2\epsilon}\psi(x_n^\pm)|f(x_n^\pm)|^\lambda, \quad 0 < \tilde{\theta}^\pm < 1.$$

Hence, using (3.9)–(3.11) for $\{x_n\} = \{x_n^+\}$ we obtain in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ that

$$\begin{aligned}
&\max\left(\left|f\left[x_n^+ - \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n^+\right]\right|,\ \left|f\left[x_n^+ + \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n^+\right]\right|\right)\\
&= \max\left(|f'(x_n^+) - \frac{1+2\epsilon}{2(1-2\epsilon)}\phi f''(\tilde{\eta}_n^+)|,\ |f'(x_n^+) + \frac{1+2\epsilon}{2(1-2\epsilon)}\phi f''(\hat{\eta}_n^+)|\right)\\
&= \begin{cases} \max\left(-f'(x_n^+) + \frac{1+2\epsilon}{2(1-2\epsilon)}\phi f''(\tilde{\eta}_n^+),\ -f'(x_n^+) - \frac{1+2\epsilon}{2(1-2\epsilon)}\phi f''(\hat{\eta}_n^+)\right), \\ \quad \text{for } f(x_n^+) < 0, \\ \max\left(f'(x_n^+) - \frac{1+2\epsilon}{2(1-2\epsilon)}\phi f''(\tilde{\eta}_n^+),\ f'(x_n^+) + \frac{1+2\epsilon}{2(1-2\epsilon)}\phi f''(\hat{\eta}_n^+)\right), \\ \quad \text{for } f(x_n^+) > 0, \end{cases}\\
&= |f'(x_n^+)| + \frac{1+2\epsilon}{2(1-2\epsilon)}\phi(x_n^+)\max(|f''(\tilde{\eta}_n^+)|,\ |f''(\hat{\eta}_n^+)|).
\end{aligned} \tag{3.12}$$

Similarly, for $\{x_n\} = \{x_n^-\}$ we obtain in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ that

$$\begin{aligned}
&\max\left(\left|f\left[x_n^- - \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n^-\right]\right|,\ \left|f\left[x_n^- + \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n^-\right]\right|\right)\\
&= |f'(x_n^-)| + \frac{1+2\epsilon}{2(1-2\epsilon)}\phi(x_n^-)\max(|f''(\tilde{\eta}_n^-)|,\ |f''(\hat{\eta}_n^-)|).
\end{aligned} \tag{3.13}$$

Thus, the expressions (3.12) and (3.13) concerning $\{x_n^+\}$ and $\{x_n^-\}$ can jointly be written in the form

$$\begin{aligned}
&\max\left(\left|f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|,\ \left|f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|\right)\\
&= |f'(x_n)| + \frac{1+2\epsilon}{2(1-2\epsilon)}\phi(x_n)\max(|f''(\tilde{\eta}_n)|,\ |f''(\hat{\eta}_n)|),
\end{aligned} \tag{3.14}$$

in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$. Hence, in view of (2.3), (3.14) and $\lambda \in (0, 1)$ we have in a δ_0-neighborhood of the simple real zero $\bar{x}$ of $f(x)$ that

$$q_1^*(x_n) = |f'(x_n)| + \frac{1+2\epsilon}{2(1-2\epsilon)}\phi(x_n)\max(|f''(\tilde{\eta}_n)|,\ |f''(\hat{\eta}_n)|),$$

$$\text{when } |x_n - \bar{x}| \leqslant \delta_0. \tag{3.15}$$

Substitution of (3.15) into (3.8) yields for $k = 1$:

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^{1+\lambda}} = \frac{(1+2\epsilon)\psi(\bar{x})|f''(\bar{x})|}{2(1-2\epsilon)|f'(\bar{x})|^{1-\lambda}}. \tag{3.16}$$

For $2 \leqslant k$ we can from $f(x) \in C^{k+1}(\mathbb{R}^1)$ write $f(x)$ as

$$f(x) = (x - \bar{x})g(x), \qquad g(\bar{x}) = \frac{f^{(k)}(x)}{k!} \neq 0. \tag{3.17}$$

Since $\lambda \in (0, 1)$ we have using (2.5) and (3.17):

$$\begin{cases} \dfrac{|f(x_n)|}{\phi(x_n)} = \dfrac{|g(x_n)|^{1-\lambda}}{\psi(x_n)} |e_n|^{(1-\lambda)k}, \\ f\left[x_n \pm \dfrac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right] = \dfrac{f^{(k)}(t_n^{\pm})}{(k-1)!}\left(1 \pm \dfrac{1+2\epsilon}{1-2\epsilon}\bar{\theta}\dfrac{\phi}{e_n}\right)^{k-1} e_n^{k-1}, \end{cases} \tag{3.18}$$

where

$$\begin{cases} t^{+} = \bar{x} + \bar{\bar{\theta}}\left(e_n + \dfrac{1+2\epsilon}{1-2\epsilon}\bar{\theta}\phi\right), \quad 0 < \bar{\bar{\theta}} < 1, \\ t^{-} = \bar{x} + \bar{\bar{\theta}}\left(e_n - \dfrac{1+2\epsilon}{1-2\epsilon}\bar{\theta}\phi\right), \quad 0 < \bar{\bar{\theta}} < 1, \\ \dfrac{\phi(x_n)}{e_n} = \psi(x_n)|g(x_n)|^{\lambda}\dfrac{|e_n|}{e_n}|e_n|^{k\lambda-1}. \end{cases} \tag{3.19}$$

Hence, we obtain

$$\begin{aligned} & q_1^*(x_n) \\ &= |e_n|^{k-1} \max\left(\frac{|g(x_n)|^{1-\lambda}}{\psi(x_n)}|e_n|^{1-\lambda k},\ \left|\frac{f^{(k)}(t_n^-)}{(k-1)!}\left(1 - \bar{\theta}\frac{1+2\epsilon}{1-2\epsilon}\frac{\phi}{e_n}\right)^{k-1}\right|,\right. \\ &\qquad \left.\left|\frac{f^{(k)}(t_n^+)}{(k-1)!}\left(1 + \bar{\theta}\frac{1+2\epsilon}{1-2\epsilon}\frac{\phi}{e_n}\right)^{k-1}\right|\right). \end{aligned} \tag{3.20}$$

Substituting (3.20) in the iterative formula (2.27) and using (2.9) directly gives

$$\begin{aligned} & \lim_{n \to \infty} \frac{e_{n+1}}{e_n} \\ &= \begin{cases} 1 - \dfrac{1}{\max\left(\dfrac{|g(\bar{x})|}{\psi(\bar{x})},\ k\left|1 + \dfrac{1+2\epsilon}{2(1-2\epsilon)}\psi(\bar{x})|g(\bar{x})|^{\lambda}\right|^{k-1}\right)}, & \text{for } k\lambda = 1, \\ 1, & \text{for } k\lambda \neq 1. \end{cases} \end{aligned} \tag{3.21}$$

(II) When $\lambda = 0$, we obtain $\phi(x_n) = \psi(x_n) > 0$. From (2.5) we have in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ that

$$q_1^*(x_n) = \max\left(\left|f'\left(x_n - \bar{\theta}\frac{1+2\epsilon}{1-2\epsilon}\psi(x_n)\right)\right|,\ \left|f'\left(x_n + \bar{\theta}\frac{1+2\epsilon}{1-2\epsilon}\psi(x_n)\right)\right|\right). \tag{3.22}$$

When $1 < \lambda$, we have in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ that

$$q_1^*(x_n) = \frac{|f(x_n)|^{1-\lambda}}{\psi(x_n)}, \quad 1-\lambda < 0. \tag{3.23}$$

Using (2.9) and substituting (3.22) and (3.23) into (2.27) gives for $1 \leqslant k$, respectively,

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \begin{cases} 1 - \dfrac{|f'(\bar{x})|}{\max\left(\left|f'\left(\bar{x} - \dfrac{1+2\epsilon}{2(1-2\epsilon)}\psi(\bar{x})\right)\right|, \left|f'\left(\bar{x} + \dfrac{1+2\epsilon}{2(1-2\epsilon)}\psi(\bar{x})\right)\right|\right)}, \\ \quad \text{for } k=1, \\ 1, \quad \text{for } 2 \leqslant k, \end{cases} \tag{3.24}$$

and

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = 1. \tag{3.25}$$

(III)(a) Since $\lambda = 1$, it follows that $\phi(x_n) = \psi(x_n)|f(x_n)|$. Using (3.14) and the hypothesis (3.5) we obtain in a δ_0-neighborhood of the simple real zero $\bar{x}$ of $f(x)$ that

$$\begin{aligned} q_1^*(x_n) &= \max\left(\frac{1}{\psi(x_n)}, \left|f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon}\phi, x_n\right]\right|, \left|f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon}\phi, x_n\right]\right|\right) \\ &= |f'(x_n)| + \frac{1+2\epsilon}{1-2\epsilon}\psi(x_n)|f(x_n)| \max(|f''(\bar{\eta}_n)|, |f''(\hat{\eta}_n)|). \end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.8) gives for $k = 1$:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = \frac{f''(\bar{x})}{2f'(\bar{x})} - (\pm 1)\frac{1+2\epsilon}{2(1-2\epsilon)}\psi(\bar{x})|f''(\bar{x})|. \tag{3.27}$$

Using (3.18) for $2 \leqslant k$ we obtain in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ that

$$\begin{aligned} &q_1^*(x_n) \\ &= \max\left(\frac{1}{\psi(x_n)}, \left|\frac{f^{(k)}(t_n^+)}{(k-1)!}\left(e_n - \bar{\theta}\frac{1+2\epsilon}{1-2\epsilon}\phi\right)^{k-1}\right|, \left|\frac{f^{(k)}(t_n^+)}{(k-1)!}\left(e_n + \bar{\theta}\frac{1+2\epsilon}{1-2\epsilon}\phi\right)^{k-1}\right|\right) \\ &= \begin{cases} |e_n|^{k-1} \cos\left(\left|\dfrac{f^{(k)}(t_n^-)}{(k-1)!}\left(1 - \bar{\theta}\dfrac{1+2\epsilon}{1-2\epsilon}\dfrac{\phi}{e_n}\right)^{k-1}\right|, \right. \\ \qquad \left. \left|\dfrac{f^{(k)}(t_n^+)}{(k-1)!}\left(1 + \bar{\theta}\dfrac{1+2\epsilon}{1-2\epsilon}\dfrac{\phi}{e_n}\right)^{k-1}\right|\right), \\ \quad \text{when (3.5) holds,} \\ 1/\psi(x_n), \\ \quad \text{when (3.5) does not hold,} \end{cases} \end{aligned} \tag{3.28}$$

where $\phi(x_n)/e_n$, t_n^- and t_n^+ are obtained by setting $\lambda = 1$ in (3.19). Substituting (3.28) in (2.27) and using (2.9) yields for $2 \leqslant k$:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \begin{cases} 1 - \dfrac{1}{k}, & \text{when (3.5) holds,} \\ 1, & \text{when (3.5) does not hold.} \end{cases} \tag{3.29}$$

(III)(b) Evidently, under condition (3.6) it follows from $\lambda = 1$ that

$$q_1^*(x_n) = \frac{1}{\psi(x_n)}, \tag{3.30}$$

in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$. Substitution of (3.30) into (2.27) gives for $1 \leqslant k$:

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \begin{cases} 1 - \psi(\bar{x})\,|\,f'(\bar{x})\,|, & \text{for } k = 1, \\ 1, & \text{for } 2 \leqslant k. \end{cases} \tag{3.31}$$

The proof is complete. □

Theorem 3.4. *Assume that the parametric function $\phi(x)$ is defined by (2.3), $f'(t)$ is monotone on the intervals $x_n - \phi(x_n) \leqslant t \leqslant x_n$ and $x_n \leqslant t \leqslant x_n + \phi(x_n)$ when $|\,x_n - \bar{x}\,| < \phi(x_n)$ and $f(x) \in C^{k+1}(\mathbb{R}^1)$, where k ($\geqslant 1$) is the multiplicity of the real zero $\bar{x}$ of $f(x)$. Taking $\hat{q}(x_n) = \hat{q}_1(x_n)$, then the following conclusions for the sequence $\{x_n\}$ generated by the iterative formula (2.26) for different values of λ hold.*

(I) *Assume $\lambda \in [0, 1)$ in $\phi(x)$; then the following hold.*

(a) *If we have in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ that*

$$|\,f'(x_n)\,| \geqslant \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\ x_n\right]\right|, \tag{3.32}$$

then the order of convergence of $\{x_n\}$ is 2 for $k = 1$, and 1 for $2 \leqslant k$. Moreover, their asymptotic error constants are

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = \frac{f''(\bar{x})}{2f'(\bar{x})}, \quad \text{for } k = 1, \tag{3.33}$$

and

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = 1 - \frac{1}{k}, \quad \text{for } 2 \leqslant k, \tag{3.34}$$

respectively.

(b) *If we have in a δ_0-neighborhood of the real zero $\bar{x}$ of $f(x)$ that*

$$|\,f'(x_n)\,| < \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi(x_n),\ x_n\right]\right|, \tag{3.35}$$

then when $\lambda \in (0, 1)$, *the order of convergence of* $\{x_n\}$ *is* $1+\lambda$ *for* $k=1$, *and* 1 *for* $2 \leqslant k$. *Their asymptotic error constants are* (3.16) *and*

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \begin{cases} 1, & \text{for } k\lambda \neq 1, \\ 1 - \dfrac{1}{\max\left(\dfrac{|g(\bar{x})|^{-\lambda}}{\phi(\bar{x})}, k\left|1 - \dfrac{1+2\epsilon}{2(1-2\epsilon)}\psi(\bar{x})|g(\bar{x})|^{\lambda}\right|^{k-1}\right)}, & \text{for } k\lambda = 1, \end{cases} \tag{3.36}$$

respectively. Also, the order of convergence of $\{x_n\}$ *is* 1 *for* $1 \leqslant k$, *when* $\lambda = 0$. *And we have*

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \begin{cases} 1 - \dfrac{|f'(\bar{x})|}{\left|f'\left(\bar{x} \pm \dfrac{1+2\epsilon}{2(1-2\epsilon)}\psi(\bar{x})\right)\right|}, & \text{for } k=1, \\ 1, & \text{for } 2 \leqslant k. \end{cases} \tag{3.37}$$

(II) *Assume* $\lambda = 1$ *in* $\phi(x)$; *then the following hold.*

(a) *If we have in a* δ_0*-neighborhood of the real zero* $\bar{x}$ *of* $f(x)$ *that*

$$\frac{1}{\psi(x_n)} \leqslant \max\left(|f'(x_n)|, \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n \right] \right| \right), \tag{3.38}$$

then the order of convergence of $\{x_n\}$ *is* 2 *for* $k=1$, *and* 1 *for* $2 \leqslant k$, *when*

$$|f'(x_n)| \geqslant \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi, x_n \right] \right|.$$

Their asymptotic error constants agree with (3.33) *and* (3.34), *respectively; and the order of convergence of* $\{x_n\}$ *is also* 2 *for* $k=1$, *and also* 1 *for* $2 \leqslant k$, *when*

$$|f'(x_n)| < \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n \right] \right|,$$

while their asymptotic error constants will agree with (3.27) *and* (3.29), *respectively.*

(b) *If we have in a* δ_0*-neighborhood of the real zero* $\bar{x}$ *of* $f(x)$ *that*

$$\frac{1}{\psi(x_n)} > \max\left(|f'(x_n)|, \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n \right] \right| \right), \tag{3.39}$$

then the order of convergence of $\{x_n\}$ *is* 1 *for* $1 \leqslant k$. *Its asymptotic error constant will agree with* (3.31).

(III) *Assume* $1 < \lambda$ *in* $\phi(x)$. *Then the order of the convergence of* $\{x_n\}$ *is* 1 *for* $1 \leqslant k$, *and we have*

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = 1.$$

Since the proof of this theorem is similar to the one of Theorem 3.3, the proof may be omitted.

When taking $q^*(x_n) = q_2^*(x_n)$ or $q_3^*(x_n)$, we have, using the proof similar to the proof of Theorem 3.3, the following theorem.

Theorem 3.5. *Assume that the parametric function* $\phi(x)$ *is defined by* (2.3), $f'(t)$ *is monotone on the interval* $|t - x_n| \leqslant \phi(x_n)$ *when* $|x_n - \bar{x}| < \phi(x)$ *and* $f(x) \in C^{k+1}(\mathbb{R}^1)$, *where* k ($\geqslant 1$) *is the multiplicity of the real zero* $\bar{x}$ *of* $f(x)$. *Then the following hold.*

(I) *If we take* $q^*(x_n) = q_2^*(x_n)$, *then the conclusions of Theorem* 3.3 *hold for the iterative sequence* $\{x_n\}$ *generated by* (2.27) *when* $0 \leqslant \lambda$ *with* $\lambda \neq 1$. *This sequence* $\{x_n\}$ *is only linearly convergent for* $1 \leqslant k$ *when* $\lambda = 1$.

(II) *If we take* $q^*(x_n) = q_3^*(x_n)$, *then the order of convergence of* $\{x_n\}$ *generated by* (2.27) *is* 1 *for* $1 \leqslant k$ *when* $0 \leqslant \lambda$.

Similarly, if we take $\hat{q}(x_n) = \hat{q}_2(x_n)$ or $\hat{q}_3(x_n)$, we have, using the proof similar to the proof of Theorem 3.3, the following theorem.

Theorem 3.6. *Assume that the parametric function* $\phi(x)$ *is defined by* (2.3), $f'(t)$ *is monotone on the intervals* $x_n - \phi(x_n) \leqslant t \leqslant x_n$ *and* $x_n \leqslant t \leqslant x_n + \phi(x_n)$ *when* $|x_n - \bar{x}| < \phi(x_n)$ *and* $f(x) \in C^{k+1}(\mathbb{R}^1)$, *where* k ($\geqslant 1$) *is the multiplicity of the real zero* $\bar{x}$ *of* $f(x)$, *then we have for the sequence* $\{x_n\}$ *generated by the iterative formula* (2.26) *for different values of* λ *that the following holds.*

(I) *If we take* $\hat{q}(x_n) = \hat{q}_2(x_n)$, *then the conclusions in Theorem* 3.4 *hold for the sequence* $\{x_n\}$ *when* $0 \leqslant \lambda$ *with* $\lambda \neq 1$, *while the order of convergence of* $\{x_n\}$ *is* 1 *for* $1 \leqslant k$ *when* $\lambda = 1$.

(II) *If we take* $\hat{q}(x_n) = \hat{q}_3(x_n)$, *then the order of convergence of* $\{x_n\}$ *is* 1 *for* $1 \leqslant k$ *when* $0 \leqslant \lambda$.

4. Numerical results

In the following examples we take

$$\phi(x) = T\left(\frac{|f(x)|}{\nu + |f(x)|}\right)^{\lambda}, \tag{4.1}$$

in view of the previous analyses in Theorems 3.3, 3.4 and Remark 2.2, where $\psi(x)$ in (2.3) is $\psi(x) = T/(\nu + |f(x)|)^{\lambda}$ with $\lambda = 0.9$ and $\nu = 2$, and we choose T an undetermined parameter. Choosing the initial parameter T_0 in the range $0.1 \leqslant T_0 \leqslant 2$ is more suitable, in general. The notation $\Delta = |x_n - x_{n-1}| < 10^{-6}$ denotes the given accuracy finding the roots of $f(x)$.

In order to lessen the iterative degree of the approximate roots satisfying the accuracy, we change the parameter T in $\phi(x)$ using the following way with less computing work. We first take the initial $T_0 = 0.8$. Next, if $|f(x_n)| \geqslant 1$, we take $T = T_0$ in order to compute $q_1^*(x_n)$ or $\hat{q}_1(x_n)$, otherwise, we choose $T = 1$, and if it causes the inequality

$$\frac{|f(x_n)|}{\phi(x_n)} \leqslant \max\left(\left|f\left[x_n - \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|,\ \left|f\left[x_n + \frac{1+2\epsilon}{1-2\epsilon}\phi,\ x_n\right]\right|\right), \tag{4.2}$$

Table 1

Formula (2.26), $\hat{q}(x_n) = \hat{q}_1(x_n)$, $\bar{\Delta} < 10^{-6}$, $\lambda = 0.9$, $\nu = 2$, $\epsilon = \frac{1}{2}(\sqrt{5} - 2)$		
$\{x_n^+\}$, $x_0 = -7$	$\{x_n^+\}$, $x_0 = 0.2$	$\{x_n^-\}$, $x_0 = 10$
$x_1 = -6.459200$	$x_1 = 0.3702804$	$x_1 = 9.671191$
$x_2 = -6.140965$	$x_2 = 0.4568426$	$x_2 = 9.281546$
$\vdots$	$\vdots$	$\vdots$
$x_{25} = -5.838841 \cdot 10^{-8}$	$x_5 = 0.5054816$	$x_{22} = 1.511543$
$x_{26} = -6.818412 \cdot 10^{-15}$	$x_6 = 0.5054823$	$x_{23} = 1.511543$
Formula (2.27), $q^*(x_n) = q_1^*(x)$, $\bar{\Delta} < 10^{-6}$, $\lambda = 0.9$, $\nu = 2$, $\epsilon = \frac{1}{2}(\sqrt{5} - 2)$		
$\{x_n^+\}$, $x_0 = -7$	$\{x_n^+\}$, $x_0 = 0.2$	$\{x_n^-\}$, $x_0 = 10$
$x_1 = -6.406243$	$x_1 = 0.3702804$	$x_1 = 9.311563$
$x_2 = -5.788492$	$x_2 = 0.4568426$	$x_2 = 8.621402$
$\vdots$	$\vdots$	$\vdots$
$x_{32} = -1.937799 \cdot 10^{-6}$	$x_{17} = 0.5054811$	$x_{29} = 1.511545$
$x_{33} = -9.689035 \cdot 10^{-7}$	$x_{18} = 0.5054816$	$x_{30} = 1.511544$
Halley's method $x_{n+1} = x_n - \dfrac{f(x_n)f'(x_n)}{f'(x_n)^2 - \frac{1}{2}f(x_n)f''(x_n)}$, $\bar{\Delta} < 10^{-6}$		
$\{x_n\}$, $x_0 = -7$	$\{x_n\}$, $x_0 = 0.2$	$\{x_n\}$, $x_0 = 10$
Still oscillatory after 1000 times	$x_1 = 0.1137927$	Still oscillatory after 1300 times
	$x_2 = 1.245597$	
	$\vdots$	
	$x_4 = 2.324573 \cdot 10^{-15}$	
	$x_5 = 0$	

or

$$\frac{|f(x_n)|}{\phi(x_n)} \leqslant \max\left(|f'(x_n)|, \left| f\left[x_n \pm \frac{1+2\epsilon}{1-2\epsilon}\phi, x_n \right] \right| \right) \tag{4.3}$$

holds when $|f(x_n)| < 1$, then T is kept, otherwise, that is, (4.2) or (4.3) does not hold, T is repeatedly replaced by mT such that (4.2) concerning (2.27) or (4.3) concerning (2.26) hold in order to compute $q_1^*(x_n)$ or $\hat{q}_1(x_n)$, where m is a suitable positive integer. In general, choosing m in the set $\{2, 3, 4, 5, 6, 7\}$ is more suitable. We take $m = 4$ here.

Example 4.1. Find all the real roots of the equation

$$f(x) = 2\sin(x^2) - x = 0, \quad x \in \mathbb{R}^1.$$

Note that the expression $f'(x_n)^2 - f(x_n)f''(x_n)$ will change sign in the iterative process when the initial value $x_0 < -6$ or $x_0 > 5$. Thus, it is inapplicable to use any iterative formula containing the radical $\{f'(x_n)^2 - f(x_n)f''(x_n)\}^{1/2}$ with global convergence to this equation for the initial values as above. On the other hand, when taking initial values $x_0 = 0.2$, -7 and 10, the numerical results obtained by the iterative formulas (2.26), (2.27) and Halley's method in [1] are displayed in Table 1.

Table 2

Formula (2.26), $\hat{q}(x_n)=\hat{q}_1(x_n)$, $\bar{\Delta}<10^{-6}$, $\lambda=0.9$, $\nu=2$, $\epsilon=\frac{1}{2}(\sqrt{5}-2)$	
$\{x_n^+\}$, $x_0=-0.5$	$\{x_n^-\}$, $x_0=14$
$x_1=-0.3196195$	$x_1=13.20000$
$x_2=-0.2841085$	$x_2=12.40000$
$\vdots$	$\vdots$
$x_{49}=-9.142097\cdot10^{-6}$	$x_{55}=8.672282\cdot10^{-6}$
$x_{50}=-8.589458\cdot10^{-6}$	$x_{56}=8.050171\cdot10^{-6}$
Formula (2.27), $q^*(x_n)=q_1^*(x_n)$, $\bar{\Delta}<10^{-6}$, $\lambda=0.9$, $\nu=2$, $\epsilon=\frac{1}{2}(\sqrt{5}-2)$	
$\{x_n^+\}$, $x_0=-0.5$	$\{x_n^-\}$, $x_0=14$
$x_1=-0.4662344$	$x_1=13.20000$
$x_2=-0.4372594$	$x_2=12.40000$
$\vdots$	$\vdots$
$x_{136}=-1.234276\cdot10^{-5}$	$x_{198}=9.685578\cdot10^{-6}$
$x_{137}=-1.148277\cdot10^{-5}$	$x_{199}=8.983464\cdot10^{-6}$
Halley's method $x_{n+1}=x_n-\dfrac{f(x_n)f'(x_n)}{f'(x_n)^2-\frac{1}{2}f(x_n)f''(x_n)}$, $\bar{\Delta}<10^{-6}$	
$\{x_n\}$, $x_0=-0.5$	$\{x_n\}$, $x_0=14$
$x_1=0.2809320$	Still oscillatory
$x_2=-0.1590819$	after 1300 times
$\vdots$	
$x_{21}=-9.176249\cdot10^{-6}$	
$x_{22}=-8.418321\cdot10^{-6}$	

Example 4.2. Find all the real roots of the equation

$$f(x)=(x-8\cdot10^{-6})(x+8\cdot10^{-6})(\sin x+2)\left((x-10)^2+18^2\right)=0,\quad x\in\mathbb{R}'.$$

Evidently, this equation has two complex conjugate roots $10+18\mathrm{i}$ and $10-18\mathrm{i}$, where $\mathrm{i}^2=-1$. Since it does not belong to the class of the entire analytic functions involving only real zeros, the iterative formulas with global convergence in [4–8] cannot be applied to this equation.

Taking initial values $x_0=-0.5$ and 14, the numerical results obtained by the iterative formulas (2.26), (2.27) and Halley's method are displayed in Table 2.

The numerical results of Tables 1 and 2 verified that the iterative methods (2.26) and (2.27) in this paper are correct, globally convergent and better than the methods in [1–3,5–7,9].

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